COHOMOLOGY BOUNDS FOR SHEAVES OF DIMENSION ONE ON $\mathbb{P}^2$

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Abstract. We find sharp bounds of $h^0(F)$ for one-dimensional semistable sheaves F on $\mathbb{P}^2$. The result generalizes the Clifford theorem. We study the stratification of the moduli space by the spectrum of sheaves. We show that the deepest stratum is isomorphic to a subscheme of a relative Hilbert scheme. This provides an example of a family of semistable sheaves having the biggest dimensional global section space.

1. Introduction and the results of the paper

1.1. **Motivations and the main theorem.** In the study of moduli spaces of semistable sheaves on the projective space, it is useful to know the upper bounds for the dimensions of the cohomology groups of the semistable sheaves with fixed the Hilbert polynomial. This is essential to the classification of semistable sheaves with respect to the cohomological conditions (For example, see [4, 9, 10] and [5]). It is also very helpful for analyzing the forgetting map from the moduli space of pairs to that of semistable sheaves (For definitions and examples, see [3, §4]).

Historically, C. Simpson constructed the moduli spaces of semistable sheaves as a compactification of the moduli space of Higgs bundles on a variety ([15]). Moduli space of Higgs bundles also has been studied by algebraic geometers and physicists regarding Hamitonian systems etc. As a natural generalization, we may consider the *twisted* Higgs bundles by using a general line bundle instead of the cotangent bundle. The moduli spaces of twisted Higgs bundles has been studied widely for its geometric structure ([14, 12]). In this paper we follow Le Potier's viewpoint where one identifies one-dimensional sheaves on $\mathbb{P}^2$ with twisted Higgs bundles on $\mathbb{P}^1$ twisted by $\mathcal{O}_{\mathbb{P}^1}(1)$.

The *spectrum* of a one-dimensional sheaf on $\mathbb{P}^2$ is the sequence of degrees in a decomposition of the corresponding Higgs bundle into line bundles on $\mathbb{P}^1$ (Definition 2.3). We study the stratification of the moduli space of semistable sheaves with respect to the spectrum. By classifying all possible spectra, we prove a conjecture on the cohomology bounds of the sheaves suggested in [4]. More precisely, let $\mathbf{M}(d,\chi)$ be the moduli space of semistable

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sheaves on $\mathbb{P}^2$ with Hilbert polynomial $dm+\chi$. Let $g(d) := \frac{(d-1)(d-2)}{2}$ be the arithmetic genus of the degree d plane curve. We prove that

Theorem 1.1. *Let F be a semistable sheaf in $\mathbf{M}(d,\chi)$.*

1. *If $\chi \geq g(d)$, then $h^0(F) = \chi$.*
2. *Suppose $\chi < g(d)$ and write $\chi + \frac{d(d-3)}{2} = kd + r$ with $0 \leq r < d$. Then,*

$$h^0(F) \leq \frac{(k+2)(k+1)}{2} + \max\{0, k-d+r+2\}. \tag{1.1}$$

Furthermore, there are families of semistable sheaves in $\mathbf{M}(d,\chi)$ for which the equality holds in (1.1)*.*

As a direct corollary, we prove the "generalized Clifford theorem" conjectured in [4]:

Corollary 1.2. [4, Conjecture 1.4] *For a sheaf F in $\mathbf{M}(d,\chi)$ with $0 \leq \chi < d$ and $h^1(F) > 0$, we have*

$$h^0(F) \leq 1 + \frac{\chi}{2} + \frac{d(d-3)}{4}.$$

The proof of Theorem 1.1 goes as follows. Part (1) is established in Lemma 2.1. For part (2), we associate a semistable Higgs bundle $(G, G \xrightarrow{\phi} G(1))$ on $\mathbb{P}^1$ (in the sense of $\mathcal{O}_{\mathbb{P}^1}(1)$-twisting) with a semistable sheaf F on $\mathbb{P}^2$, where $G = \pi_* F$ is the direct image sheaf of F by the projection $\pi : \mathbb{P}^2 - \{a\} \to \mathbb{P}^1$ from a point a ($\notin$ Supp(F)). Since F is pure, G is locally free and hence it decomposes into a direct sum of line bundles on $\mathbb{P}^1$. For the sheaf F to be stable, the degrees of these line bundles must satisfy certain numerical conditions (Proposition 2.4), which enables us to determine the upper bounds of $h^0(F)$ by a combinatorial reasoning (Lemma 2.7).

We study the stratification of the moduli space by the spectrum of sheaves. It turns out that our choice of the spectrum that determines the upper bounds of $h^0(F)$ corresponds to the deepest stratum in this stratification, in the sense that it has the biggest codimension among all strata. We identify this stratum with a subscheme of a relative Hilbert scheme. In particular, this proves the bounds in Theorem 1.1 are sharp bounds.

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2. The proof of Theorem 1.1

The following is from [1, Lemma 4.2.4].

Lemma 2.1. *Let F be a semistable sheaf on $\mathbb{P}^2$ with Hilbert polynomial $dm+\chi$. Then $H^1(F) = 0$ if $\chi \geq g(d)$.*

Proof. We know that F is supported on some degree e Cohen-Macaulay curve C in $\mathbb{P}^2$ where $1 \le e \le d$. By adjunction formula, we have $\omega_C \simeq \mathcal{O}_C(e-3)$. By applying Serre duality on C, we have $H^1(F)^* \simeq \mathrm{Hom}(F, \mathcal{O}_C(e-3))$. Suppose there is a nonzero map: $F \to \mathcal{O}_C(e-3)$. Then, by semistability of F and $\mathcal{O}_C$, we have

$$\mu(F) \le \mu(\mathcal{O}_C(e-3)). \tag{2.1}$$

Since the Hilbert polynomial $\chi(\mathcal{O}_C(e-3)(m)) = em + \frac{e(e-3)}{2}$, we have $\mu(\mathcal{O}_C(e-3)) = \frac{e-3}{2}$.

Then by (2.1),

$$\frac{\chi}{d} \le \frac{e-3}{2} \le \frac{d-3}{2}.$$

Therefore, if $\chi \ge \frac{d(d-3)}{2} + 1 = \frac{(d-1)(d-2)}{2}$, then $H^1(F) = 0$. □

To prove main theorem, let us recall the notion of the *spectrum of a pure sheaf* [7]. Let $\pi : \mathbb{P}^2 - \{a\} \to \mathbb{P}^1$ be the projection map from a point $a \in \mathbb{P}^2$. For a pure sheaf F with $a \notin \mathrm{Supp}(F)$, the direct image sheaf $G := \pi_* F$ is a locally free sheaf on $\mathbb{P}^1$. Note that $\mathbb{P}^2 - \{a\} = \mathrm{Tot}(\mathcal{O}_{\mathbb{P}^1}(1)) = \mathrm{Spec}(\mathrm{Sym}\mathcal{O}_{\mathbb{P}^1}(-1))$ and π is the affine morphism $\mathrm{Spec}(\mathrm{Sym}\mathcal{O}_{\mathbb{P}^1}(-1)) \to \mathbb{P}^1$. Hence, the locally free sheaf G on $\mathbb{P}^1$ has a natural $\mathcal{O}_{\mathbb{P}^1}(-1)$-module structure or equivalently, it admits a sheaf homomorphism $\phi : G \to G(1)$.

Remark 2.2. The pair $(G, G \xrightarrow{\phi} G(1))$ is a *twisted Higgs bundle* on $\mathbb{P}^1$. In [7], it is shown that a sheaf is (semi)stable if and only if the associated twisted Higgs bundle is (semi)stable. Let U_a be the open subscheme of $\mathbf{M}(d,\chi)$ consisting of sheaves whose support does not pass through a fixed point $a \in \mathbb{P}^2$ (cf. [7]). It follows that U_a is isomorphic to the moduli space of twisted Higgs bundles.

Definition 2.3. Under the above notations and assumptions,

$$G = \pi_* F \cong \oplus_{i=1}^{d} \mathcal{O}_{\mathbb{P}^1}(a_i)$$

is a locally free sheaf on $\mathbb{P}^1$ of rank d if the Hilbert polynomial of F is $\chi(F(m)) = dm + \chi$. Let us define the degree sequence

$$\nu = [a_1, a_2, \cdots, a_d]$$

where $a_1 \ge \cdots \ge a_d$ *the spectrum of the sheaf* F.

Proposition 2.4. *Let* $\nu = [a_1, a_2, \cdots, a_d]$ *be the spectrum of a semistable sheaf* F *in* $\mathbf{M}(d,\chi)$. *Then it must satisfy two conditions:*

(1) $\sum_{i=1}^{d} a_i = \chi - d$ *and*
(2) $a_j - a_{j+1} \le 1$ *for all* $1 \le j \le d-1$ *(*Balanced property*).*

Proof. Part (1) is obvious from the condition $\chi(F) = \chi$. Part (2) is equivalent to the semistability of F, see the proof of [7, Lemma 3.12]. □

Note that if $\pi_*F = \oplus_{i=1}^d \mathcal{O}_{\mathbb{P}^1}(a_i)$,

$$h^0(F) := \dim H^0(\mathbb{P}^2, F) = \sum_{a_k \geq 0} (a_k + 1)$$

since the map π is an affine map.

Remark 2.5. To determine $[a_1, \cdots, a_d]$, it is enough to know $k \mapsto h^0(F(k))$ or equivalently $k \mapsto h^1(F(k))$. In particular, the spectrum of a sheaf is independent of the choice of the center of the projection π.

Example 2.6.
(1) Let C be a curve of degree d in $\mathbb{P}^2$. Then the spectrum of the structure sheaf $\mathcal{O}_C$ is

$$[0, -1, -2, \cdots, 1-d].$$

Indeed, for $0 \leq k \leq d-1$, we have $h^0(\mathcal{O}_C(k)) = \frac{(k+1)(k+2)}{2}$ and the above spectrum is the unique one satisfying this condition.
(2) Let $[a_1, \cdots, a_d]$ be the spectrum of a semistable sheaf F with multiplicity d. Then the spectrum of $F(k)$ is $[a_1 + k, \cdots, a_d + k]$ and the spectrum of $F^D := \mathcal{E}xt^1(F, \omega_{\mathbb{P}^2})$ is $[-2 - a_d, \cdots, -2 - a_1]$. In fact, the spectrum of $F(k)$ is straightforward by the projection formula $\pi_*F(k) = \pi_*F \otimes \mathcal{O}_{\mathbb{P}^1}(k)$. For F^D, by [11, Proposition 5] or [1, Proposition 4.2.8], we have $h^0(F^D(-k)) = h^1(F(k))$ for any integer k. By a straightforward induction, one can prove the claim.
(3) If a spectrum ν satisfies the conditions in Proposition 2.4, there exists a semistable sheaf whose spectrum is ν. In fact, we can explicitly construct examples of *torus equivariant* semistable sheaves in each spectrum by the method of [2]. (cf. Example 2.9)

Lemma 2.7. *Let* $[a_1, \cdots, a_d]$ *be the spectrum of* F *in* $\mathbf{M}(d, \chi)$*. If* $a_j = a_{j+1}$ *for at most one* j*, then* $h^0(F)$ *is maximal among all sheaves in* $\mathbf{M}(d, \chi)$*.*

Proof. Suppose not. Let $[a_1, \cdots, a_d]$ be the spectrum for a sheaf F in $\mathbf{M}(d, \chi)$ having bigger $h^0(F)$. Suppose first that we have $a_{j-1} > a_j = a_{j+1} = \cdots = a_{j+k} > a_{j+k+1}$ for some j and $k \geq 2$. Consider the spectrum

$$[\cdots, a_{j-1}, a_j + 1, a_{j+1}, \cdots, a_{j+k-1}, a_{j+k} - 1, a_{j+k+1}, \cdots].$$

It is easy to see that the Euler characteristic does not change while $h^0(F)$ either increases or remains the same. Hence we may assume the same number is not repeated more than twice in $[a_1, \cdots, a_d]$.

Now we suppose that $a_j = a_{j+1}$ for at least two j. Take $a_{j_1} = a_{j_1+1}$ and $a_{j_2} = a_{j_2+1}$, $j_1 < j_2$. We can choose $[a_1, \cdots, a_d]$ so that $j_2 - j_1$ is minimal. Consider the spectrum

$$[a_1, \cdots, a_{j_1} + 1, a_{j_1+1}, \cdots, a_{j_2}, a_{j_2+1} - 1, \cdots, a_d].$$

This spectrum has the same Euler characteristic, but $h^0(F)$ either increases or remains the same. This contradicts the minimality of $j_2 - j_1$. $\square$

As we will see in the following examples, when we fix d and χ, the spectrum of a sheaf whose global section space is maximal need not be unique.

Example 2.8. (1) There is unique spectrum satisfying the condition in Lemma 2.7 for a given (d, χ). We present here a few examples of such spectra for the convenience of readers.

(d, χ)	
$(5, 0)$	$[1, 0, -1, -2, -3]$
$(5, 1)$	$[1, 0, -1, -2, -2]$
$(5, 2)$	$[1, 0, -1, -1, -2]$
$(5, 3)$	$[1, 0, 0, -1, -2]$
$(6, 3)$	$[2, 1, 0, -1, -2, -3]$
$(6, 4)$	$[2, 1, 0, -1, -2, -2]$
$(6, 5)$	$[2, 1, 0, -1, -1, -2]$
$(6, 6)$	$[2, 1, 0, 0, -1, -2]$

When (d, χ) is of the form $(2k + 1, 0)$ or $(2k, k)$, the corresponding spectrum is

$$[k - 1, k - 2, \cdots, k - d].$$

If (d, χ) is of the form $(2k + 1, s)$ or $(2k, k + s)$ for $0 \leq s \leq d$, last s terms are increased by one from $[k - 1, k - 2, \cdots, k - d]$. Similar rules can be found for $s > d$.

(2) Note that the converse of Lemma 2.7 is not true, that is, there may be different types of spectrum having the largest global section space. For example, one can check that there are three possible spectra for $(d, \chi) = (6, 0)$:

$$\{[0, 0, 0, -1, -2, -3], \ [1, 0, -1, -1, -2, -3], \ [1, 0, -1, -2, -2, -2]\}.$$

For the corresponding loci of semistable sheaves in $\mathbf{M}(6, 0)$, see the Table 4 of [10]. We remark that the locus corresponding to the spectrum $[1, 0, -1, -1, -2, -3]$ has the biggest codimension. We will focus on such loci in §3.

Returning to our main claim, we prove the part (2) of Theorem 1.1.

Proof of Theorem 1.1. It remains to compute $h^0(F)$ of the sheaf F having the spectrum satisfying the condition in Lemma 2.7. We will show the existence of such sheaves later in Example 2.9 and §3.

Such spectrum is completely determined by a_1 and $1 \leq j \leq d$ such that $a_j = a_{j+1}$. Here, $j = d$ means there is no j with $a_j = a_{j+1}$. For notational convenience, we let $k := a_1$ and $r := d - j$, $0 \leq r \leq d - 1$.

Then it is easy to see that

$$a_1 + \cdots + a_d = kd - \frac{d(d-1)}{2} + r. \tag{2.2}$$

So, $\chi = kd - \frac{d(d-3)}{2} + r$ and hence k and r are uniquely determined by χ.

Suppose $k \geq d-3$ and $r \geq 1$, in other words, $\chi \geq \frac{d(d-3)}{2}+1 = g(d)$. Then we see that $a_d \geq -1$, which implies that all higher cohomologies vanish and we have $h^0(F) = \chi$. This gives another proof of Lemma 2.1.

Now we suppose $\chi < g(d)$. Then the spectrum is given by

$$[k, k-1, \cdots, k-d+r+1, k-d+r+1, \cdots, k-d+2].$$

If $r \leq d-k-1$, the nonnegative terms in the spectrum are $(k, k-1, \cdots, 0)$ and hence we have

$$h^0(F) = \frac{(k+2)(k+1)}{2}.$$

On the other hand, if $r > d-k-1$ we have

$$h^0(F) = \frac{(k+2)(k+1)}{2} + (k-d+r+2).$$

So the theorem follows. □

The following example shows that the bounds in Theorem 1.1 are sharp.

Example 2.9. Let k and r be determined by d and χ as in the proof of Theorem 1.1. Let us denote by x, y, and z the homogeneous coordinates for $\mathbb{P}^2$. Let C_d be the d-fold thickening of a fixed line in $\mathbb{P}^2$. For example, Let C_d be the subscheme defined by the ideal $\langle z^d \rangle$.

Let Z_r be the subscheme of C_d defined by the ideal $\langle x, z^{d-r} \rangle$. We take $F = I_{Z_r, C_d}(k+1)$ be the twisted ideal sheaf of Z_r in C_d. Then F is a semistable sheaf in $\mathbf{M}(d,\chi)$ with $h^0(F)$ equal to the bound in Theorem 1.1. (cf. [1, §2.3], [2])

Corollary 2.10. *Suppose $0 \leq \chi < d$. Let F be a semistable sheaf in $\mathbf{M}(d,\chi)$. Then*

$$h^0(F) \leq \begin{cases} \frac{d^2-1}{8}, & \textit{if } d \textit{ is odd and } \chi < \frac{d}{2}, \\ \frac{d^2-4d+3}{8} + \chi, & \textit{if } d \textit{ is odd and } \chi > \frac{d}{2}, \\ \frac{d(d-2)}{8} + \chi, & \textit{if } d \textit{ is even and } \chi < \frac{d}{2}, \\ \frac{d(d+2)}{8}, & \textit{if } d \textit{ is even and } \chi \geq \frac{d}{2}. \end{cases}$$

Proof. This is a consequence of an elementary calculation from Theorem 1.1. □

In particular, this proves Corollary 1.2.

3. Stratification of $\mathbf{M}(d,\chi)$ via spectra

In this section, we study the stratification of $\mathbf{M}(d,\chi)$ with respect to the spectrum. It turns out that the spectrum we chose in §2 corresponds to the closed stratum with the biggest codimension (cf. Proposition 3.2). In Proposition 3.9 and Proposition 3.11, we describe this locus in terms of relative Hilbert schemes by using the wall-crossing technique of [3].

Let ν be a spectrum. We denote by $\mathbf{M}_\nu(d,\chi)$ the locus in $\mathbf{M}(d,\chi)$ consisting of sheaves having spectrum ν.

Lemma 3.1. $\{\mathbf{M}_\nu(d,\chi)\}$ *is a finite locally closed stratification of* $\mathbf{M}(d,\chi)$.

Proof. As noted in Remark 2.5, the spectrum of a sheaf F is completely determined by $k \mapsto h^0(F(k))$. Hence the subscheme $\mathbf{M}_\nu(d,\chi)$ is cut out by conditions on $h^0(F(k))$. By the semicontinuity theorem, it is clear that $\mathbf{M}_\nu(d,\chi)$ is locally closed. Since only finitely many spectra are possible after fixing d and χ, $\mathbf{M}_\nu(d,\chi)$ is a finite locally closed stratification of $\mathbf{M}(d,\chi)$. $\square$

The strata provided by our classification are irreducible varieties.

Proposition 3.2. $\mathbf{M}_\nu(d,\chi)$ *is an irreducible variety of* $\mathbf{M}(d,\chi)$ *whose codimension is the size of the set* $\{(a_i,a_j)\colon |a_i-a_j|\geq 2\}$.

Proof. Fix a point $a \in \mathbb{P}^2$. As in §2, let $U_a = \{F \in \mathbf{M}_\nu(d,\chi)\colon a \notin \mathrm{Supp}(F)\}$. We first claim that it is enough to show that $U_{a,\nu} := \mathbf{M}_\nu(d,\chi) \cap U_a$ is an irreducible variety. Consider $R_\nu = \{(F,p) \in \mathbf{M}_\nu(d,\chi) \times \mathbb{P}^2\colon p \notin F\}$. Then the projection $R_\nu \to \mathbf{M}_\nu(d,\chi)$ is surjective. We also have the surjective morphism

$$\begin{array}{ccc} \mathrm{PGL}_3 \times U_{a,\nu} & \to & R_\nu \\ (\alpha, F) & \mapsto & (\alpha^* F, \alpha^{-1} a) \end{array}.$$

where $\alpha \in \mathrm{PGL}_3$ is considered as an automorphism $\alpha\colon \mathbb{P}^2 \to \mathbb{P}^2$. Hence $\mathbf{M}(d,\chi)$ is irreducible provided that $U_{a,\nu}$ is irreducible.[1] The semistability is an open condition ([7]) and thus the locus of semistable Higgs bundles is an open subset of the affine space $\mathrm{Hom}(G, G(1))$ with the spectrum $\nu = G$. Hence from the quotient map

$$\mathrm{Hom}(G, G(1))^{ss} \to U_{a,\nu},$$

one can say that $U_{a,\nu}$ is an irreducible variety.

Secondly, the codimension of $\mathbf{M}_\nu(d,\chi)$ in $\mathbf{M}(d,\chi)$ directly comes from the proof of Proposition 3.14 in [7]. $\square$

Example 3.3. In the series of papers [4, 9, 10], Maican et al have studied the moduli spaces $\mathbf{M}(d,\chi)$ for $d \leq 6$ using a stratification. They have classified sheaves F in $\mathbf{M}(d,\chi)$ by conditions on $h^i(F(j))$ and $h^0(F \otimes \Omega^1(1))$, which in turn determine the syzygy types of the sheaves F. In many cases, their stratification coincides with our stratification by spectra. For example, $\mathbf{M}(4,1)$ is a union of X_0 and X_1 in [4]. In our notation, X_0 is $\mathbf{M}_{[0,-1,-1,-1]}(4,1)$ and X_1 is $\mathbf{M}_{[1,0,-1,-1]}(4,1)$. However, since they have used the additional conditions on $h^0(F \otimes \Omega^1(1))$, their stratification is finer than the stratification by spectra.

This stratification by spectra becomes very complicated as d increases. In this paper, we confine ourselves to the deepest stratum.

Definition 3.4. (1) Let $\nu = [a_1, \cdots, a_d]$ be a spectrum. We call ν the *deepest spectrum* if it satisfies the condition in Lemma 2.7.

[1] This argument is due to S. Katz.

(2) When ν is the unique deepest spectrum for fixed d and χ, we denote $\mathbf{M}_\nu(d,\chi)$ by $\mathcal{C}_{d,\chi}$.

Proposition 3.5. *$\mathcal{C}_{d,\chi}$ is the unique closed stratum in $\{\mathbf{M}_\nu(d,\chi)\}$ having the biggest codimension among strata.*

Proof. We note that $\nu = [a_1, \cdots, a_d]$ is the deepest spectrum if and only if a_1 is maximal and a_d is minimal among all spectra for $\mathbf{M}(d,\chi)$. Hence we have

$$\mathcal{C}_{d,\chi} = \{G \in \mathbf{M}(d,\chi) \colon h^0(G(-a_1)) > 0 \text{ and } h^1(G(-a_d-2)) > 0\}.$$

Thus, it is closed by the semicontinuity theorem. Moreover, it is clear from Proposition 3.2 that $\mathcal{C}_{d,\chi}$ has the biggest codimension. □

Hence we see that $\mathcal{C}_{d,\chi}$ is indeed the deepest stratum in the stratification $\{\mathbf{M}_\nu(d,\chi)\}$.

Proposition 3.6. (1) $\mathcal{C}_{d,\chi} \simeq \mathcal{C}_{d,d+\chi}$.
(2) *If $0 \le \chi < d$, then $\mathcal{C}_{d,\chi} \simeq \mathcal{C}_{d,d-\chi}$.*

Proof. The first statement is obtained by twisting by $\mathcal{O}_{\mathbb{P}^2}(1)$. For the second statement, by part (2) of Example 2.6, under the isomorphism $\mathbf{M}(d,\chi) \simeq \mathbf{M}(d, d-\chi)$ sending F to $F^D(1)$, sheaves in $\mathcal{C}_{d,\chi}$ correspond to sheaves in $\mathcal{C}_{d,d-\chi}$. □

Hence it is enough to consider only finitely many cases for fixed d. In what follows, we describe $\mathcal{C}_{d,\chi}$ as a subscheme of the relative Hilbert scheme. We start with reviewing results in [3].

Definition 3.7. We denote by $\mathbf{B}(d,n)$ the relative Hilbert scheme consisting of pairs (C,Z) where C is degree d curve in $\mathbb{P}^2$ and $Z \subset C$ is length n subscheme.

In [3], we study the relationship between $\mathbf{B}(d,n)$ and the moduli space of stable sheaves using α-stable pairs. A pair (s,F) consists of a sheaf F on $\mathbb{P}^2$ and one-dimensional subspace $s \subset H^0(F)$. Given a positive number α, a pair (s,F) is called *α-semistable* if F is pure and for any subsheaves $F' \subset F$, the inequality

$$\frac{\chi(F'(m)) + \delta\cdot\alpha}{r(F')} \le \frac{\chi(F(m)) + \alpha}{r(F)}$$

holds for $m \gg 0$. Here $r(F)$ is the leading coefficient of Hilbert polynomial $\chi(F(m))$ and $\delta = 1$ if the section s factors through F' and $\delta = 0$ otherwise. When the strict inequality holds, (s,F) is called α-stable. By the work of Le Potier [8], there exist the moduli spaces $\mathbf{M}^\alpha(d,\chi)$ which parameterizes S-equivalent classes[2] of α-semistable pairs with Hilbert polynomial $dm+\chi$.

At the extreme values of α, the moduli spaces of α-stable pairs are related to $\mathbf{B}(d,n)$ and $\mathbf{M}(d,\chi)$.

[2]By definition, two semistable pairs are S-equivalent if they have isomorphic Jordan-Hölder filtration.

(1) If the stability parameter α is sufficiently large (denoted by $\alpha = \infty$), this moduli space $\mathbf{M}^{\alpha}(d, \chi)$ is isomorphic to the relative Hilbert scheme $\mathbf{B}(d, n)$ where n is $\chi - 1 + g(d)$. The correspondence is the following [13, Proposition B.8]: For ∞-stable pair (s, F), the section $s : \mathcal{O}_{\mathbb{P}^2} \to F$ induces a short exact sequence

$$0 \to \mathcal{O}_C \to F \to Q \to 0$$

for a degree d curve C and zero dimensional sheaf Q with length n. As taking dual $\mathcal{H}om_C(-, \mathcal{O}_C)$ to the short exact sequence, we obtain a zero dimensional subscheme Z defined by the surjection:

$$\mathcal{O}_C \twoheadrightarrow \mathcal{E}xt^1_C(Q, \mathcal{O}_C),$$

where the ideal sheaf is given by $I_{Z,C} = \mathcal{H}om_C(F, \mathcal{O}_C)$.

(2) On the other hand, if the stability parameter is sufficiently small (denoted by $\alpha = 0^+$), then there exists a forgetting morphism into the space $\mathbf{M}(d, \chi)$, denoted by

$$\xi : \mathbf{M}^{0^+}(d, \chi) \to \mathbf{M}(d, \chi).$$

The wall-crossing behavior of the moduli spaces $\mathbf{M}^{\alpha}(d, \chi)$ is studied in [6, 3]. The moduli space changes only finitely many values of α, which are called *walls*. The wall occurs if there exist strictly α-semistable pairs. If there is no wall between $\mathbf{M}^{\infty}(d, \chi)$ and $\mathbf{M}^{0^+}(d, \chi)$, two spaces are isomorphic.

Remark 3.8. Let us denote by $\mathbf{M}^{\alpha}_{\nu}(d, \chi)$ the subscheme of $\mathbf{M}^{\alpha}(d, \chi)$ consisting of pairs (s, F) where the sheaf F has the spectrum ν. Then one can also consider the wall-crossing behavior of $\mathbf{M}^{\alpha}_{\nu}(d, \chi)$ for a fixed ν: the spectrum of a sheaf remains unchanged after the elementary modification of pairs [3].

We also remark that strictly semistable sheaves exist in $\mathbf{M}_{\nu}(d, \chi)$ if and only if $\alpha = 0$ is a "wall" for $\mathbf{M}^{\alpha}_{\nu}(d, \chi)$, in which case the forgetting morphism ξ: $\mathbf{M}^{0^+}_{\nu}(d, \chi) \to \mathbf{M}_{\nu}(d, \chi)$ is a projective bundle. (cf. [1, Lemma 4.2.2]) Furthermore, if $h^0(F) = 1$ for any $(s, F) \in \mathbf{M}_{\nu}(d, \chi)$, then ξ induces an isomorphism $\mathbf{M}^{0^+}_{\nu}(d, \chi) \simeq \mathbf{M}_{\nu}(d, \chi)$. We will use this fact repeatedly.

Proposition 3.9. *Suppose* d *is odd and* $0 \leq \chi < \frac{d}{2}$. *Then* $\mathcal{C}_{d,\chi}$ *is isomorphic to the subscheme of* $\mathbf{B}(d, \chi)$ *consisting of pairs* (C, Z) *where* C *is a degree* d *curve and* Z *is colinear* χ *points on* C.

Proof. Let $\nu = [a_1, \cdots, a_d]$ be the corresponding deepest spectrum. From the condition, it is easy to check that $a_1 = \frac{d-3}{2}$ and $a_1 > a_2$. We also have $a_d = -\frac{d+1}{2}$ if $\chi = 0$ and $a_d = -\frac{d-1}{2}$ if $\chi > 0$. When $\chi = 0$, the same reasoning as below shows that the sheaves in $\mathcal{C}_{d,\chi}$ is of the form $\mathcal{O}_C$ for some degree d curve C. Hence $\mathcal{C}_{d,0} \simeq \mathbf{B}(d, 0)$ as required.

Assume $\chi > 0$ so that $a_d = -\frac{d-1}{2}$. Let $\chi' = \chi - d(\frac{d-3}{2})$ and $\nu' = [a_1 - \frac{d-3}{2}, \cdots, a_d - \frac{d-3}{2}]$. Then by twisting by $-\frac{d-3}{2}$ we have

$$\mathcal{C}_{d,\chi} \simeq \mathcal{C}_{d,\chi'} = \{F \in \mathbf{M}(d, \chi') \colon h^0(F) = 1 \text{ and } h^1(F(d-4)) > 0\}.$$

Therefore, $\mathcal{C}_{d,\chi} \simeq \mathbf{M}^{0^+}_{\nu'}(d,\chi')$. We claim that no wall crossing is necessary between $\mathbf{M}^{0^+}(d,\chi')$ and $\mathbf{M}^{\infty}(d,\chi')$. Indeed, if there is a wall, the splitting type of wall is

$$(1, dm+\chi') = (1,(d-e)m+1-g(d-e)+z)+(0, em+\chi'-z-1+g(d-e)),$$

for an integer $1 \le e < d$ and a nonnegative integer z. Then corresponding stability parameter α is given by

$$\frac{\chi'+\alpha}{d} = \frac{1}{e}(\chi'-z-1+g(d-e)).$$

After simplifying, we get

$$\alpha = (e-d)(\frac{d}{2}-\frac{\chi}{e}) - \frac{d}{e}z.$$

So, if $\chi < \frac{d}{2}$, α is negative for any e and z. Hence, there is no wall and we have $\mathcal{C}_{d,\chi} \simeq \mathbf{M}^{\infty}_{\nu'}(d,\chi')$.

Under isomorphism $\mathbf{M}^{\infty}(d,\chi') \cong \mathbf{B}(d,\chi)$ described above, a stable pair (s,F) in $\mathbf{M}^{\infty}(d,\chi')$ corresponds to a pair of its support C and a length χ subscheme Z of C whose structure sheaf is given by $\mathcal{E}xt^1(Q,\mathcal{O}_C)$. We recall that $I_{Z,C} = \mathcal{H}om_C(F,\mathcal{O}_C)$.

We now show that the condition $h^1(F(d-4)) > 0$ is equivalent to the condition Z being colinear. By Serre duality on C, we have

$$H^1(F(d-4))^* = \mathrm{Hom}_C(F(d-4),\mathcal{O}_C(d-3)) = \mathrm{Hom}_C(F,\mathcal{O}_C(1)).$$

Therefore, $h^1(F(d-4)) > 0$ if and only if

$$0 \ne \mathrm{Hom}_C(F,\mathcal{O}_C(1)) = H^0(I_{Z,C}(1)) = H^0(I_{Z,\mathbb{P}^2}(1)),$$

or equivalently, Z is colinear. □

Remark 3.10. For $\frac{d}{2} < \chi < d$, the same proof works except for the case $\chi = d-1$. If $\chi = d-1$, we have $a_1 = a_2$ so that $h^0(F) = 2$ for $F \in \mathbf{M}(d,\chi')$ and hence $\mathbf{M}^{0^+}_{\nu}(d,\chi') \simeq \mathbf{M}^{\infty}_{\nu}(d,\chi')$ is a $\mathbb{P}^1$-bundle over $\mathcal{C}_{d,\chi}$. Geometrically, this means a pair of $(d-1)$-tuples of colinear points is identified if d-th points determined by the line and the curve coincide. This is consistent with Proposition 3.6, that is, the subscheme of $\mathbf{B}(d,\chi)$ having colinear points is isomorphic to the subscheme of $\mathbf{B}(d,d-\chi)$ having colinear points unless $\chi = 1$ or $d-1$.

Proposition 3.11. *Suppose d is even and $\frac{d}{2} \le \chi < d$. Then $\mathcal{C}_{d,\chi}$ is isomorphic to the subscheme of $\mathbf{B}(d,\chi-\frac{d}{2})$ consisting of pairs (C,Z) where C is a degree d curve and Z is colinear $\chi-\frac{d}{2}$ points on C.*

Proof. The proof is parallel to Proposition 3.9. Note that the case $\chi = d$ is dropped, which we will show is the only case that we have to consider wall-crossing.

Let $\nu = [a_1,\cdots,a_d]$ be the corresponding deepest spectrum as before. Then we have $a_1 = \frac{d-2}{2}$ and $a_1 > a_2$. We also have $a_d = -\frac{d}{2}$ if $\chi = \frac{d}{2}$ and $a_d = -\frac{d-2}{2}$ if $\chi > \frac{d}{2}$.

The same argument as in Proposition 3.9 works with $\chi' = \chi - d(\frac{d-2}{2})$. We only have to check that no wall-crossing occurs. Similarly as before if there is a wall, the splitting type of the wall is

$$(1, dm+\chi') = (1, (d-e)m+1-g(d-e)+z)+(0, em+\chi'-z-1+g(d-e)),$$

for an integer $1 \le e < d$ and a nonnegative integer z. Then corresponding stability parameter α is given by

$$\frac{\chi'+\alpha}{d} = \frac{1}{e}(\chi' - z - 1 + g(d-e)).$$

After simplifying, we get

$$\alpha = (e-d)(d(\frac{e+1}{2e}) - \frac{\chi}{e}) - \frac{d}{e}z.$$

Since $1 \le e < d$ and $\chi < d$, α is negative. Hence, there is no wall. □

Remark 3.12. By direct computation of the dimension, one can easily check that the codimension of $\mathcal{C}_{d,\chi}$ in $\mathbf{M}(d,\chi)$ is $\frac{d^2-3d-2}{2}$, which is the largest one among all spectra. (cf. [7, Proposition 3.14])

Note that in the proof of Proposition 3.11, we have a wall $\alpha = 0$ when $\chi = d$, $e = 1$, and $z = 0$. This is because of the presence of strictly semistable sheaves.

Corollary 3.13. *Let $0 \le \chi < d$. Unless d is even and $\chi = 0$, there are no strictly semistable sheaves in $\mathcal{C}_{d,\chi}$. If d is even and $\chi = 0$, the locus $\mathcal{C}^s_{d,0}$ of stable sheaves in $\mathcal{C}_{d,0}$ is isomorphic to the subscheme of relative Hilbert scheme $\mathbf{B}(d, \frac{d}{2})$ consisting of (C, Z) where Z is colinear and the line containing Z is not a component of C.*

Proof. The strictly semistable sheaves exist if and only if $\alpha = 0$ becomes a wall. In the proof of Proposition 3.9 and 3.11, we have a wall at $\alpha = 0$ only when d is even and $\chi = 0$. Since $e = 1$ and $z = 0$ is the only possibility, a pair (s, F) in $\mathcal{C}_{d,0}$ is strictly semistable only if its support is a union of degree $(d-1)$ curve C' and a line L and the section s is taken from the structure sheaf $\mathcal{O}_{C'}$, so that the cokernel of s is supported on L. □

This corollary generalizes the description of the deepest stratum of $\mathbf{M}(4,0)$ and $\mathbf{M}(6,0)$ in [4, 10].

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